

Semilinear substructural logics with the finite embeddability property [☆]

SanMin Wang

Faculty of Science, Zhejiang Sci-Tech University, Hangzhou 310018, P.R. China

Abstract

In this paper, three semilinear substructural logics \mathbf{UL}_ω , \mathbf{IUL}_ω and \mathbf{HpsUL}_ω^* are constructed. Then the completeness of \mathbf{UL}_ω and \mathbf{IUL}_ω with respect to classes of finite \mathbf{UL} and \mathbf{IUL} -algebras, respectively, is proved. Algebraically, non-integral \mathbf{UL}_ω and \mathbf{IUL}_ω -algebras have the finite embeddability property, which gives a characterization for finite \mathbf{UL} and \mathbf{IUL} -algebras. Furthermore, the standard completeness of \mathbf{UL}_ω , \mathbf{IUL}_ω and \mathbf{HpsUL}_ω^* is proved, which shows that they are substructural fuzzy logics.

Keywords: Finite embeddability property, Residuated lattices, Semilinear substructural logics, Standard completeness, Fuzzy logic

2000 MSC: 03B52, 06F99, 03B50, 03B47

1. Introduction

The finite embeddability property (FEP), or rather, the finite model property (FMP), as shown in [16], fails for some known non-integral semilinear substructural logics including Metcalfe and Montagna's uninorm logic \mathbf{UL} and involutive uninorm logic \mathbf{IUL} [11], and a suitable extension \mathbf{HpsUL}^* [13] of Metcalfe, Olivetti and Gabbay's pseudo-uninorm logic \mathbf{HpsUL} [10]. This shows that \mathbf{UL} , \mathbf{IUL} and \mathbf{HpsUL}^* are incomplete with respect to classes of finite algebras involved.

A natural problem is whether we can construct logics which are complete with respect to finite \mathbf{UL} , \mathbf{IUL} and \mathbf{HpsUL}^* -algebras. Algebraically, our

[☆]This work is supported by the National Foundation of Natural Sciences of China (Grant No: 61379018)

Email address: wangsanmin@hotmail.com (SanMin Wang)

motivation is how to characterize the variety generated by its finite members when a class of algebras does not enjoy the FEP (or FMP).

In this paper, we construct three schematic extensions \mathbf{UL}_ω , \mathbf{IUL}_ω and \mathbf{HpsUL}_ω^* by adding one simple axiom

$$(\text{FIN}) \quad (A \odot B \rightarrow e) \leftrightarrow (A \odot B \odot B \rightarrow e)$$

to \mathbf{UL} , \mathbf{IUL} and \mathbf{HpsUL}^* , respectively. Then we prove that \mathbf{UL}_ω and \mathbf{IUL}_ω are complete with respect to classes of finite \mathbf{UL} and \mathbf{IUL} -algebras, respectively. Algebraically, non-integral \mathbf{UL}_ω and \mathbf{IUL}_ω -algebras have the finite embeddability property, which gives a characterization for finite \mathbf{UL} and \mathbf{IUL} -algebras.

Classes of \mathbf{UL}_ω and \mathbf{IUL}_ω -algebras are non-integral varieties which usually, as pointed out in [7], do not enjoy the FEP. We prove the FEP for \mathbf{UL}_ω and \mathbf{IUL}_ω -algebras by Blok and Alten's construction [1,2]. But in proving the finiteness of Blok and Alten's construction, we use specific property of \mathbf{UL}_ω and \mathbf{IUL}_ω -algebras other than that of Dicksons lemma or Higman's finite basis theorem, see Lemma 4.7.

By Wang's construction [13~15], we prove that \mathbf{UL}_ω , \mathbf{IUL}_ω and \mathbf{HpsUL}_ω^* are complete with respect to algebras whose lattice reducts are the real unit interval $[0,1]$, i.e., they are standard complete. This shows that they are substructural fuzzy logics and semilinear substructural logics [4]. In addition, we are unable to prove the FEP for \mathbf{HpsUL}_ω^* and left it as an open problem.

2. \mathbf{HpsUL}_ω^* , \mathbf{UL}_ω , \mathbf{IUL}_ω and algebras involved

The Hilbert system \mathbf{HpsUL} is the logic of bounded representable residuated lattices, which is based on a countable propositional language with formulas built inductively as usual from a set of propositional variables, binary connectives $\odot, \rightarrow, \rightsquigarrow, \wedge, \vee$ and constants e, f, \perp, \top , with definable connectives:

$$\begin{aligned} \neg\varphi &:= \varphi \rightarrow f, \\ \varphi \leftrightarrow \psi &:= (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi), \\ \varphi^n &:= \overbrace{\varphi \odot \cdots \odot \varphi}^{n \text{ times}}, \\ \lambda_\chi(\varphi) &:= (\chi \rightarrow \varphi \odot \chi) \wedge e, \\ \rho_\chi(\varphi) &:= (\chi \rightsquigarrow \chi \odot \varphi) \wedge e. \end{aligned}$$

Definition 2.1. **HpsUL** consists of the following axioms and rules [9, 10]:

- (A₁) $\vdash \varphi \rightarrow \varphi$
- (A₂) $\vdash (\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$
- (A₃) $\vdash \varphi \rightarrow ((\varphi \rightsquigarrow \psi) \rightarrow \psi)$
- (A₄) $\vdash (\varphi \rightsquigarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightsquigarrow \chi))$
- (A₅) $\vdash \psi \rightarrow (\varphi \rightarrow \varphi \odot \psi)$
- (A₆) $\vdash (\psi \rightarrow (\varphi \rightarrow \chi)) \rightarrow (\varphi \odot \psi \rightarrow \chi)$
- (A₇) $\vdash (\psi \rightsquigarrow \psi \odot (\psi \rightarrow \varphi)) \rightarrow (\psi \rightsquigarrow \varphi)$
- (A₈) $\vdash (\varphi \wedge t) \odot (\psi \wedge t) \rightarrow \varphi \wedge \psi$
- (A₉) $\vdash \varphi \wedge \psi \rightarrow \psi$
- (A₁₀) $\vdash \varphi \wedge \psi \rightarrow \varphi$
- (A₁₁) $\vdash (\chi \rightarrow \varphi) \wedge (\chi \rightarrow \psi) \rightarrow (\chi \rightarrow \varphi \wedge \psi)$
- (A₁₂) $\vdash \varphi \rightarrow \varphi \vee \psi$
- (A₁₃) $\vdash \psi \rightarrow \varphi \vee \psi$
- (A₁₄) $\vdash (\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi) \rightarrow (\varphi \vee \psi \rightarrow \chi)$
- (A₁₅) $\vdash e$
- (A₁₆) $\vdash \varphi \rightarrow (e \rightarrow \varphi)$
- (A₁₇) $\vdash \varphi \rightarrow \top$
- (A₁₈) $\vdash \perp \rightarrow \varphi$
- (PRL) $\vdash (\lambda_\chi(\varphi \vee \psi \rightarrow \varphi)) \vee (\rho_\chi(\varphi \vee \psi \rightarrow \psi))$
- (MP) $\varphi, \varphi \rightarrow \psi \vdash \psi$
- (ADJ_U) $\varphi \vdash \varphi \wedge e$
- (PN _{\rightarrow}) $\varphi \vdash \psi \rightarrow \varphi \odot \psi$
- (PN _{\rightsquigarrow}) $\varphi \vdash \psi \rightsquigarrow \psi \odot \varphi$

Definition 2.2. [11, 13] A logic is a schematic extension (extension for short) of **HpsUL** if it results from **HpsUL** by adding axioms in the same language.

In particular,

- **HpsUL*** is **HpsUL** plus (WCM) $\vdash (\varphi \rightsquigarrow e) \rightarrow (\varphi \rightarrow e)$;
- **UL** is **HpsUL** plus $\vdash \varphi \odot \psi \rightarrow \psi \odot \varphi$;
- **IUL** is **UL** plus $\vdash \neg\neg\varphi \rightarrow \varphi$.
- For a positive integer $n \geq 2$, **C_nHpsUL*** is **HpsUL*** plus the n-potency axiom $\vdash \varphi^n \leftrightarrow \varphi^{n-1}$.

Definition 2.3. New extensions of **HpsUL** are defined as follows.

- **HpsUL*** _{ω} is **HpsUL*** plus (FIN) $\vdash (\varphi \odot \psi \rightarrow e) \leftrightarrow (\varphi \odot \psi \odot \psi \rightarrow e)$;
- **UL** _{ω} and **IUL** _{ω} are **UL** and **IUL** plus (FIN), respectively.

Let $\mathbf{L} \in \{\mathbf{HpsUL}^*, \mathbf{UL}, \mathbf{IUL}, \mathbf{HpsUL}_\omega^*, \mathbf{UL}_\omega, \mathbf{IUL}_\omega\}$ in the remainder of this section. A proof in \mathbf{L} of a formula φ from a set Γ of formulas is defined as usual. We write $\Gamma \vdash_{\mathbf{L}} \varphi$ if such a proof exists.

Definition 2.4. [9, 10] An **HpsUL**-algebra is a bounded residuated lattice $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, \rightsquigarrow, e, f, \perp, \top \rangle$ with universe A , binary operations $\wedge, \vee, \cdot, \rightarrow, \rightsquigarrow$, and constants e, f, \perp, \top such that:

- (i) $\langle A, \wedge, \vee, \perp, \top \rangle$ is a bounded lattice with top element \top and bottom element \perp ;
- (ii) $\langle A, \cdot, e \rangle$ is a monoid;
- (iii) $\forall x, y, z \in A, x \cdot y \leq z$ iff $x \leq y \rightsquigarrow z$ iff $y \leq x \rightarrow z$;
- (iv) $\forall x, y, u, v \in A, (\lambda_u(x \vee y \rightarrow x)) \vee (\rho_v(x \vee y \rightarrow y)) = e$, where, for any $a, b \in A$, $\lambda_a(b) := (a \rightarrow b \cdot a) \wedge e$, $\rho_a(b) := (a \rightsquigarrow a \cdot b) \wedge e$.

We use the convention that \cdot binds stronger than other binary operations and we shall often omit \cdot ; we will thus write xy instead of $x \cdot y$, for example. Suitable classes of algebras of extensions of **HpsUL** are defined as follows.

Definition 2.5. [13, 11] Let $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, \rightsquigarrow, e, f, \perp, \top \rangle$ be an **HpsUL**-algebra. For \mathbf{L} an extension of **HpsUL**, \mathcal{A} is an \mathbf{L} -algebra if all axioms of \mathbf{L} are valid in \mathcal{A} . An \mathbf{L} -chain is an \mathbf{L} -algebra that is linearly ordered. In particular:

- \mathcal{A} is an **HpsUL**^{*}-algebra if the weak commutativity (Wcm) holds: $xy \leq e$ iff $yx \leq e$ for all $x, y \in A$;
- \mathcal{A} is an **UL**-algebra if $xy = yx$ for all $x, y \in A$;
- \mathcal{A} is an **IUL**-algebra if it is an **UL**-algebra such that $\neg\neg x = x$ for all $x \in A$;
- \mathcal{A} is a **C_nHpsUL**^{*}-algebra if \mathcal{A} is an **HpsUL**^{*}-algebra such that $x^n = x^{n-1}$ for all $x \in A$.
- \mathcal{A} is an **HpsUL** _{ω} ^{*}-algebra (**UL** _{ω} or **IUL** _{ω} -algebra) if it is an **HpsUL**^{*}-algebra (**UL** or **IUL**-algebra) such that the following identity (Fin) holds: $xy \rightarrow e = xy^2 \rightarrow e$ for all $x, y \in A$;

Definition 2.6. [11] Let $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, \rightsquigarrow, e, f, \perp, \top \rangle$ be an \mathbf{L} -algebra. (i) An \mathcal{A} -valuation v is a homomorphism from the term algebra determined by formulas in \mathbf{L} to \mathcal{A} ; (ii) A formula φ is valid in \mathcal{A} if $v(\varphi) \geq e$ holds for any \mathcal{A} -valuation v ; (iii) The relation of semantic consequence $\Gamma \models_{\mathcal{A}} \varphi$ holds if each \mathcal{A} -evaluation that validates all formulae in a theory Γ validates φ as well.

Theorem 2.7. [4, 11] $\Gamma \vdash_{\mathbf{L}} \varphi$ iff $\Gamma \models_{\mathcal{A}} \varphi$ for every \mathbf{L} -chain \mathcal{A} , i.e., \mathbf{L} is an implicative semilinear logic.

Theorem 2.8. [9, 12] (i) Each \mathbf{L} -algebra has a subdirect representation with \mathbf{L} -chains; (ii) each finite \mathbf{L} -algebra has a subdirect representation with finitely many finite \mathbf{L} -chains.

Lemma 2.9. Let \mathcal{A} be an $\mathbf{HpsUL}_{\omega}^*$ -algebra. Then $xy \leq e$ iff $xy^2 \leq e$ for all x, y in A .

Proof. Let $xy \leq e$ then $e \leq xy \rightarrow e$. Thus $e \leq xyy \rightarrow e$ by (Fin). Hence $xyy \leq e$. The sufficiency part of the lemma can be proved in the same way. \square

Lemma 2.10. Let \mathcal{A} be an \mathbf{HpsUL}^* -algebra such that $xy \leq e$ iff $xy^2 \leq e$ for all x, y in A . Then \mathcal{A} is an $\mathbf{HpsUL}_{\omega}^*$ -algebra.

Proof. We prove $xy \rightarrow e = xy^2 \rightarrow e$ for all x, y in A . By $xy \rightarrow e \leq xy \rightarrow e$, we get $xy(xy \rightarrow e) \leq e$. Then $(xy \rightarrow e)xy \leq e$ by (Wcm). Thus $(xy \rightarrow e)xy^2 \leq e$. Hence $xy^2(xy \rightarrow e) \leq e$ by (Wcm). Therefore $xy \rightarrow e \leq xy^2 \rightarrow e$. Similarly, we can prove $xy^2 \rightarrow e \leq xy \rightarrow e$. Thus $xy \rightarrow e = xy^2 \rightarrow e$. \square

The following properties hold in each $\mathbf{C}_n\mathbf{HpsUL}^*$ -chain [13].

Lemma 2.11. Let \mathcal{A} be an $\mathbf{HpsUL}_{\omega}^*$ -chain. Then

- (1) $stu = s$ implies $st = s$ and $su = s$;
- (2) $stu = t$ implies $st = t$ and $tu = t$;
- (3) $stu = u$ implies $su = u$ and $tu = u$;
- (4) $st = e$ implies $s = t = e$;
- (5) $st > u$ iff $t > s \rightarrow u$ iff $s > t \rightsquigarrow u$;
- (6) $e \rightarrow s = s$;
- (7) $su > tu$ implies $s > t$.

Proof. Only (1) is proved as follows and others can be proved in the same way. If $tu \leq e$ then $tut \leq e$ and $utu \leq e$ by Lemma 2.9 and (Wcm). Thus $stut \leq s$ and $stutu \leq st$. Hence $st \leq s$ and $s \leq st$. Therefore $st = s$. The case of $tu > e$ can be proved in the same way. \square

Lemma 2.12. (i) Each finite \mathbf{HpsUL}^* -chain is an $\mathbf{HpsUL}_{\omega}^*$ -chain; (ii) Each finite \mathbf{HpsUL}^* -algebra is an $\mathbf{HpsUL}_{\omega}^*$ -algebra.

Proof. (i) Let \mathcal{A} be a finite **HpsUL**^{*}-chain. We prove that $xy \rightarrow e = xy^2 \rightarrow e$ for all x, y in A . Since \mathcal{A} is finite, there is a positive integer n such that \mathcal{A} is a **C_nHpsUL**^{*}-chain. Suppose that $xy \rightarrow e > xy^2 \rightarrow e$ then $xy^2(xy \rightarrow e) > e$. Let $z = xy \rightarrow e$ then $xyz \leq e < xy^2z$. Thus $zxy \leq e < zxy^2$ by (Wcm).

If $y^k = y^{k-1}$ and $zxy \leq e < zxy^2$ for any $k \geq 3$. Then $zxy^{k-1} \leq y^{k-2} \leq zxy^k$. Thus $zxy^{k-1} = y^{k-2}$ by $y^k = y^{k-1}$. Hence $y^{k-1} = y^{k-2}$ by Lemma 2.11 (3).

Since $y^n = y^{n-1}$ and $zxy \leq e < zxy^2$, then $y^{n-1} = y^{n-2}, \dots, y^2 = y$ by repeatedly applying the property above. Thus $zxy = zxy^2$, a contradiction and hence $xy \rightarrow e \leq xy^2 \rightarrow e$. Similarly, we can prove that $xy^2 \rightarrow e \leq xy \rightarrow e$. Thus $xy \rightarrow e = xy^2 \rightarrow e$.

(ii) follows from (i) and Theorem 2.8 (ii). \square

Clearly, Lemmas 2.9~2.12 hold for all **UL** _{ω} and **IUL** _{ω} -algebras.

3. Blok and Alten's Construction for **HpsUL** _{ω} ^{*}, **UL** _{ω} , **IUL** _{ω} -algebras

Definition 3.1. Given an ordered algebra $\mathcal{A} = \langle A, \langle f_i^{\mathcal{A}} : i \in I \rangle, \leq^{\mathcal{A}} \rangle$ (of any type), with $\leq^{\mathcal{A}}$ a (partial) order on A , and any non-empty subset $B \subseteq A$, the partial subalgebra \mathcal{B} of \mathcal{A} with domain B is the ordered partial algebra $\mathcal{B} = \langle B, \langle f_i^{\mathcal{B}} : i \in I \rangle, \leq^{\mathcal{B}} \rangle$, where $a \leq^{\mathcal{B}} b$ iff $a \leq^{\mathcal{A}} b$ for all $a, b \in B$, and for each $i \in I$, $f_i^{\mathcal{B}}$ k -ary, $b_1, \dots, b_k \in B$,

$$f_i^{\mathcal{B}}(b_1, \dots, b_k) = \begin{cases} f_i^{\mathcal{A}}(b_1, \dots, b_k) & \text{if } f_i^{\mathcal{A}}(b_1, \dots, b_k) \in B, \\ \text{undefined} & \text{if } f_i^{\mathcal{A}}(b_1, \dots, b_k) \notin B. \end{cases}$$

Definition 3.2. A partial embedding of an ordered partial algebra \mathcal{B} into an ordered algebra \mathcal{A} is a one one map $\iota : B \rightarrow A$ such that (i) $a \leq^{\mathcal{B}} b$ iff $\iota(a) \leq^{\mathcal{A}} \iota(b)$ for all $a, b \in B$; (ii) $\iota(f_i^{\mathcal{B}}(b_1, \dots, b_k)) = f_i^{\mathcal{A}}(\iota(b_1), \dots, \iota(b_k))$ if $f_i^{\mathcal{B}}(b_1, \dots, b_k)$ is defined for some operation f_i and $b_1, \dots, b_k \in B$ where $f_i^{\mathcal{A}}$ denotes the realization of f_i in \mathcal{A} .

Definition 3.3. A class **K** of ordered algebras of the same type has the finite embeddability property (FEP for short) if every finite partial subalgebra \mathcal{B} of any algebra $\mathcal{A} \in \mathbf{K}$ can be partially embedded into some finite member of **K**.

Lemma 3.4. Let **K** be a variety and **K**_{si} be the class of all subdirectly irreducible members of **K**. Then **K** has the FEP if **K**_{si} has the FEP.

Proof. See Lemma 20 of [3]. \square

Definition 3.5. Let $\mathcal{A} = \langle A, \cdot, \rightarrow, \rightsquigarrow, \wedge, \vee, e, f, \perp, \top \rangle$ be an \mathbf{HpsUL}_ω^* -chain and $\mathcal{B} = \langle B, \cdot, \rightarrow, \rightsquigarrow, \wedge, \vee, e, f, \perp, \top \rangle$ be a partial subalgebra of \mathcal{A} such that $\{e, f, \perp, \top\} \subseteq B$. Let $\mathcal{M} = \langle M, \cdot, \wedge, \vee, e, f, \perp, \top \rangle$ be the linearly ordered submonoid of $\langle A, \cdot, \wedge, \vee, e, f, \perp, \top \rangle$ generated by B .

Let $a_1, \dots, a_n \in M$ and let $\delta_1, \dots, \delta_n \in \{l, r\}$ (l and r stand for “left” and “right”, respectively). We will write \mathbf{a}^δ to denote the sequence $a_1^{\delta_1} \dots a_n^{\delta_n}$, we will use ε to denote the empty sequence and we denote by $M^{l,r}$ the set of all possible \mathbf{a}^δ , that is,

$$M^{l,r} = \{a_1^{\delta_1} \dots a_n^{\delta_n} : n < \omega; a_1, \dots, a_n \in M; \delta_1, \dots, \delta_n \in \{l, r\}\}.$$

Clearly any two elements of $M^{l,r}$ can be concatenated to form a new element of $M^{l,r}$. The sequence \mathbf{a}^δ is to be understood as a unary polynomial operating on M , defined inductively as follows: For each $c \in M$, set $\varepsilon(c) = c$ and, for $\mathbf{a}^\delta \in M^{l,r}$ and $b \in M$, set $\mathbf{a}^\delta b^l(c) = \mathbf{a}^\delta(b \cdot c)$ and $\mathbf{a}^\delta b^r(c) = \mathbf{a}^\delta(c \cdot b)$.

For each $\mathbf{a}^\delta \in M^{l,r}$ and $b \in B$, define

$$(\mathbf{a}^\delta)^{-1}(b) = \{c \in M : \mathbf{a}^\delta(c) \leq b\}, (b) = \{c \in M : c \leq b\},$$

$$\bar{D} = \{(\mathbf{a}^\delta)^{-1}(b) : \mathbf{a}^\delta \in M^{l,r}, b \in B\}, D = \{\bigcap \chi : \chi \subseteq \bar{D}\}.$$

For $X \subseteq M$, define

$$C(X) = \bigcap \{(\mathbf{a}^\delta)^{-1}(b) \in \bar{D} : X \subseteq (\mathbf{a}^\delta)^{-1}(b)\}.$$

For $X, Y \subseteq M$ and $X_i \subseteq M, i \in I$, define

$$XY = \{ab : a \in X, b \in Y\}, Xa = X \{a\}, X \cdot^D Y = C(XY),$$

$$X \rightarrow^D Y = \{a \in M : Xa \subseteq Y\}, X \rightsquigarrow^D Y = \{a \in M : aX \subseteq Y\},$$

$$\bigvee_{i \in I}^D X_i = C(\cup_{i \in I} X_i), \bigwedge_{i \in I}^D X_i = \bigcap_{i \in I} X_i, \sim X = X \rightarrow^D (f),$$

$$\perp^D = (\perp) = \{\perp\}, \top^D = (\top) = M, e^D = (e), f^D = (f).$$

When \mathcal{A} is an \mathbf{HpsUL}_ω^* -chain, we may define for all $a_l, a_r \in M, b \in B$,

$$(\mathbf{a}^\delta)^{-1}(b) = \{c \in M : a_l c a_r \leq b\}.$$

When \mathcal{A} is an \mathbf{UL}_ω -chain, we may simplify $(\mathbf{a}^\delta)^{-1}(b)$ as

$$(a \mapsto b) = \{c \in M : ac \leq b\}$$

for all $a \in M, b \in B$.

Lemma 3.6. *If \mathcal{A} is an \mathbf{HpsUL}_ω^* -chain. Then the following properties hold.*

- (1) $\{\perp^D, \top^D, t^D, f^D\} \subseteq \bar{D}$ and $C(X) = X$ for all $X \in D$;
- (2) $X \subseteq C(X), C(X) \subseteq C(Y)$ if $X \subseteq Y$ and $C(C(X)) = C(X)$ for all $X, Y \subseteq M$;
- (3) $X \vee^D Y \rightarrow^D Z = (X \rightarrow^D Z) \wedge^D (Y \rightarrow^D Z)$;
- (4) If $X \subseteq M$ and $Y_i \subseteq M$ for $i \in I$, then $X \rightarrow^D \bigcap_{i \in I} Y_i = \bigcap_{i \in I} (X \rightarrow^D Y_i)$ and $X \rightsquigarrow^D \bigcap_{i \in I} Y_i = \bigcap_{i \in I} (X \rightsquigarrow^D Y_i)$;
- (5) If $X \subseteq M$ and $Y \in D$ then $X \rightarrow^D Y \in D$;
- (6) $X \cdot^D e^D = e^D \cdot^D X = X, (X \cdot^D Y) \cdot^D Z = X \cdot^D (Y \cdot^D Z) = C(XYZ)$ for all $X, Y, Z \in D$ and, $X \cdot^D Y \subseteq e^D$ iff $Y \cdot^D X \subseteq e^D$ for all $X, Y \in D$;
- (7) $X \cdot^D Y \subseteq Z$ iff $Y \subseteq X \rightarrow^D Z$ iff $X \subseteq Y \rightsquigarrow^D Z$ for all $X, Y, Z \in D$;
- (8) $X \rightarrow^D (Y \rightarrow^D Z) = Y \cdot^D X \rightarrow^D Z$ for all $X, Y \subseteq M$ and $Z \in D$;
- (9) $\sim \sim \sim X = \sim X$ for all $X \subseteq M$;
- (10) If $a, b \in B$ and $a \rightarrow b \in B$ then $(a \mapsto b) = (a] \rightarrow^D (b]$, where, (9) and (10) are valid if \mathcal{A} is an \mathbf{UL}_ω (or \mathbf{IUL}_ω)-chain.

Proof. See Section 5 of [1] and Section 2 of [2]. □

Lemma 3.7. *for all $X, Y, U, V \in D$,*

$$e^D = (\lambda_U(X \vee^D Y \rightarrow^D X)) \vee^D (\rho_V(X \vee^D Y \rightarrow^D Y)).$$

Proof. Let $z \in e^D$ then $z \leq e$. Suppose that

$$z \notin (\lambda_U(X \vee^D Y \rightarrow^D X)) \vee^D (\rho_V(X \vee^D Y \rightarrow^D Y)).$$

Then

$$\begin{aligned} z &\notin (U \rightarrow^D (X \vee^D Y \rightarrow^D X) \cdot^D U) \wedge^D e^D, \\ z &\notin (V \rightsquigarrow^D V \cdot^D (X \vee^D Y \rightarrow^D Y)) \wedge^D e^D \end{aligned}$$

by Lemma 3.6(2). Thus

$$\begin{aligned} z &\notin U \rightarrow^D (X \vee^D Y \rightarrow^D X) \cdot^D U, \\ z &\notin (V \rightsquigarrow^D V \cdot^D (X \vee^D Y \rightarrow^D Y)) \end{aligned}$$

by $z \in e^D$. Hence there exist $u \in U, v \in V$ such that

$$uz \notin (X \vee^D Y \rightarrow^D X) \cdot^D U, \quad zv \notin V \cdot^D (X \vee^D Y \rightarrow^D Y).$$

Since D is downward closed, then

$$u \notin (X \vee^D Y \rightarrow^D X) \cdot^D U, \quad v \notin V \cdot^D (X \vee^D Y \rightarrow^D Y)$$

by $uz \leq u, zv \leq v$. Thus

$$u \notin (X \vee^D Y \rightarrow^D X) \cdot U, \quad v \notin V \cdot (X \vee^D Y \rightarrow^D Y).$$

Hence $e \notin X \vee^D Y \rightarrow^D X$ and $e \notin X \vee^D Y \rightarrow^D Y$. Therefore

$$e \notin (X \rightarrow^D X) \wedge^D (Y \rightarrow^D X), \quad e \notin (X \rightarrow^D Y) \wedge^D (Y \rightarrow^D Y)$$

by Lemma 3.6(3). Then $e \notin Y \rightarrow^D X, e \notin X \rightarrow^D Y$ by $e \in X \rightarrow^D X, e \in Y \rightarrow^D Y$. Thus there exist $z' \in Y$ such that $z' \notin X$ and $z'' \in X, z'' \notin Y$.

Suppose that $z' \leq z''$ then $z' \in X$, since X is downward closed and $z'' \in X$, a contradiction with $z' \notin X$. Suppose that $z'' \leq z'$ then $z'' \in Y$, since Y is downward closed and $z' \in Y$, a contradiction with $z'' \notin Y$. Then $z' \not\leq z''$ and $z'' \not\leq z'$, which contradicts with the linearity of M . Hence

$$z \in (\lambda_U(X \vee^D Y \rightarrow^D X)) \vee^D (\rho_V(X \vee^D Y \rightarrow^D Y)).$$

Then

$$e^D \subseteq (\lambda_U(X \vee^D Y \rightarrow^D X)) \vee^D (\rho_V(X \vee^D Y \rightarrow^D Y)).$$

Since

$$(U \rightarrow^D (X \vee^D Y \rightarrow^D X) \cdot^D U) \wedge^D e^D \subseteq e^D$$

and

$$(V \rightarrow^D V \cdot^D (X \vee^D Y \rightarrow^D Y)) \wedge^D e^D \subseteq e^D,$$

then

$$(\lambda_U(X \vee^D Y \rightarrow^D X)) \cup (\rho_V(X \vee^D Y \rightarrow^D Y)) \subseteq e^D.$$

Thus

$$(\lambda_U(X \vee^D Y \rightarrow^D X)) \vee^D (\rho_V(X \vee^D Y \rightarrow^D Y)) \subseteq e^D.$$

Then

$$e^D = (\lambda_U(X \vee^D Y \rightarrow^D X)) \vee^D (\rho_V(X \vee^D Y \rightarrow^D Y)).$$

□

Lemma 3.8. $X \cdot^D Y \subseteq (e]$ iff $X \cdot^D Y \cdot^D Y \subseteq (e]$ for all $X, Y \in D$.

Proof. Let $X \cdot^D Y \subseteq (e]$. Then $C(XY) \subseteq (e]$. Thus $XY \subseteq (e]$. Hence $xy \leq e$ for all $x \in X, y \in Y$. Let $x \in X, y, y' \in Y$ then $xy \leq e$ and $xy' \leq e$. Thus $yx \leq e$ by (Wcm). Then $yxxy' \leq e$. Hence $y'yxx \leq e$ by (Wcm). Thus $y'yx \leq e$ by Lemma 2.9. Therefore $xy'y \leq e$ by (Wcm). Thus $XY Y \subseteq (e]$. Then $C(XYY) \subseteq (e]$ by Lemma 3.6(2) and $C((e]) = (e]$. Hence $X \cdot^D Y \cdot^D Y \subseteq (e]$ by Lemma 3.6(6).

Let $X \cdot^D Y \cdot^D Y \subseteq (e]$. Then $C(XYY) \subseteq (e]$ by Lemma 3.6(6). Thus $XY Y \subseteq (e]$. Let $x \in X, y \in Y$ then $xyy \leq e$. Thus $xy \leq e$ by Lemma 2.9. Hence $XY \subseteq (e]$. Therefore $C(XY) \subseteq (e]$ by Lemma 3.6(2) and $C((e]) = (e]$. Then $X \cdot^D Y \subseteq (e]$. \square

Lemma 3.9. (i) $\mathcal{D} = \langle D, \cdot^D, \rightarrow^D, \rightsquigarrow^D, \vee^D, \wedge^D, e^D, f^D, \perp^D, \top^D \rangle$ is an \mathbf{HpsUL}_ω^* -algebra if \mathcal{A} is an \mathbf{HpsUL}_ω^* -chain;

(ii) $\mathcal{D} = \langle D, \cdot^D, \rightarrow^D, \vee^D, \wedge^D, e^D, \perp^D, \top^D \rangle$ is an \mathbf{UL}_ω -algebra if \mathcal{A} is an \mathbf{UL}_ω -chain;

(iii) $\mathcal{D} = \langle D, \vee^D, \wedge^D, \perp^D, \top^D \rangle$ is a complete lattice;

(iv) $\bigwedge_{i \in I}^D X_i \rightarrow^D Y = \bigvee_{i \in I}^D (X_i \rightarrow^D Y)$ and $\bigvee_{i \in I}^D X_i \rightarrow^D Y = \bigwedge_{i \in I}^D (X_i \rightarrow^D Y)$.

Proof. (i) and (ii) are immediately from Lemma 3.6(6) and 3.6(7), Lemma 3.7 and 3.8. (iii) is clear. (iv) follows from (i), (ii) and (iii). \square

Lemma 3.10. Let \mathcal{A} be a linearly ordered \mathbf{IUL}_ω -algebra and \mathcal{B} be a partial subalgebra of \mathcal{A} such that $\{e, f, \perp, \top\} \subseteq B$ and $\neg b \in B$ for all $b \in B$. Then

(i) $[b] = \sim \sim [b]$;

(ii) $(a \mapsto b) = \sim \sim (a \mapsto b)$ for all $(a \mapsto b) \in \bar{D}$;

(iii) $X = \sim \sim X$ for all $X \in D$;

(iv) $\mathcal{D} = \langle D, \cdot^D, \rightarrow^D, \vee^D, \wedge^D, e^D, f^D, \perp^D, \top^D \rangle$ is an \mathbf{IUL}_ω -algebra.

Proof. (i) Let $b \in B$ then $\sim \sim [b] = \sim [b] \rightarrow^D (f) = (([b] \rightarrow^D (f)) \rightarrow^D (f) = (\neg b] \rightarrow^D (f) = (\neg b \rightarrow f) = [b]$ by Lemma 3.6(10) and $b, f, \neg b \in B$. Thus $\sim \sim [b] = [b]$.

(ii) Let $(a \mapsto b) \in \bar{D}$. Then $(a \mapsto b) = \{a\} \rightarrow^D (b) = \{a\} \rightarrow^D \sim \sim (b) = \{a\} \rightarrow^D (\sim (b) \rightarrow^D (f)) = \sim (b) \cdot^D \{a\} \rightarrow^D (f) = \sim (\sim (b) \cdot^D \{a\})$ by (i) and Lemma 3.6(8). Thus $(a \mapsto b) = \sim (\sim (b) \cdot^D \{a\})$. Hence $\sim \sim (a \mapsto b) = \sim \sim \sim (\sim (b) \cdot^D \{a\}) = \sim (\sim (b) \cdot^D \{a\}) = (a \mapsto b)$ by Lemma 3.6(9).

(iii) Let $X \in D$. Then we can write $X = \bigcap_{i \in I} (a_i \mapsto b_i) = \bigwedge_{i \in I}^D (a_i \mapsto b_i)$.

Then

$$\begin{aligned}
\sim\sim X &= (X \rightarrow^D (f]) \rightarrow^D (f] \\
&= (\bigwedge_{i \in I}^D (a_i \mapsto b_i] \rightarrow^D (f]) \rightarrow^D (f] \\
&= \bigvee_{i \in I}^D ((a_i \mapsto b_i] \rightarrow^D (f]) \rightarrow^D (f] \\
&= \bigwedge_{i \in I}^D (((a_i \mapsto b_i] \rightarrow^D (f]) \rightarrow^D (f]) \\
&= \bigwedge_{i \in I}^D (a_i \mapsto b_i] = X.
\end{aligned}$$

by (ii) and Lemma 3.9 (iv).

(iv) is immediately from (iii) and Lemma 3.9 (ii). \square

Lemma 3.11. *The map $\iota : \mathcal{B} \rightarrow \mathcal{D}$, which sends a to $(a] = \{x \in M : x \leq a\}$ for $a \in B$, is an partial embedding of the partial subalgebra \mathcal{B} of \mathcal{A} into \mathcal{D} . Moreover, $\iota(e) = e^D$, $\iota(f) = f^D$, $\iota(\perp) = \perp^D$, $\iota(\top) = \top^D$ and ι preserves all meets and joins that exist in \mathcal{B} .*

Proof. It is proved by a procedure similar to that of Lemma 2.6 of [2]. \square

4. Finite embeddability property and decidability

In this section we show that \mathbf{UL}_ω and \mathbf{IUL}_ω have the finite embeddability property and are hence decidable.

Let \mathbb{Z}_+ denote the set of positive integers, $\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$. We sometimes write $p_1 \cdots p_k$ by $\prod_{i=1}^k p_i$ for simplicity. We denote the i -th component of $\alpha = (m_1, \dots, m_k) \in \mathbb{N}^k$ by $\alpha(i)$, i.e., $\alpha(i) = m_i$ for all $1 \leq i \leq k$.

Definition 4.1. A subsequence index is a mapping $\sigma : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ such that $n \leq \sigma(n) < \sigma(n+1)$ for all n in \mathbb{Z}_+ .

Remark 4.2. There is a correspondence between the set of subsequences of a sequence and the set of subsequence indexes, i.e., (i) let $\{\alpha_n\}$ be a sequence and σ be a subsequence index then $\{\alpha_{\sigma(n)}\}$ is subsequences of $\{\alpha_n\}$; (ii) there is a subsequence index σ for each subsequences $\{\alpha_{n_l}\}$ of $\{\alpha_n\}$ such that $\sigma(l) = n_l$ for all l in \mathbb{Z}_+ .

Definition 4.3. Let $k \in \mathbb{Z}_+$ and $\{\alpha_n\}$ be a sequence in \mathbb{N}^k . $\{\alpha_n\}$ is an Ω -sequence if $\{\alpha_n(i)\}$ is an infinite constant chain or an infinite ascending chain for all $1 \leq i \leq k$. A subsequence index σ is an Ω -subsequence index if $\{\alpha_{\sigma(n)}\}$ is an Ω -subsequence of $\{\alpha_n\}$.

Lemma 4.4. (i) Let $\{\alpha_n\}$ be a sequence in \mathbb{N} then there exists a subsequence index σ such that $\{\alpha_{\sigma(n)}\}$ is an Ω -subsequence of $\{\alpha_n\}$;

(ii) The composition $\sigma_1 \circ \sigma_2$ of two subsequence indexes σ_1 and σ_2 of $\{\alpha_n\}$ is a subsequence index of $\{\alpha_n\}$;

(iii) $\sigma_2 \circ \sigma_1$ is an Ω -subsequence index of $\{\alpha_n\}$ if σ_2 is an Ω -subsequence index of $\{\alpha_n\}$ and σ_1 is a subsequence index of $\{\alpha_n\}$.

Proof. (i) $\{\alpha_n\}$ contains an infinite constant subsequence if it is bounded. Otherwise, it contains an infinite ascending subsequence. Then it contains an Ω -subsequence.

(ii) That is to say, the subsequence of any subsequence of $\{\alpha_n\}$ is a subsequence of $\{\alpha_n\}$.

(iii) That is to say, the subsequence $\{\alpha_{\sigma_2 \circ \sigma_1(n)}\}$ of the Ω -subsequence $\{\alpha_{\sigma_1(n)}\}$ is an Ω -subsequence. \square

Lemma 4.5. Let $\{\alpha_n\}$ be a sequence in \mathbb{N}^k . Then there exists a subsequence index σ such that $\{\alpha_{\sigma(n)}\}$ is an Ω -subsequence of $\{\alpha_n\}$.

Proof. Sequentially, we construct subsequence indexes $\sigma_1, \sigma_2, \dots, \sigma_k$ such that $\{\alpha_{\sigma_1(n)}(1)\}$, $\{\alpha_{\sigma_1 \circ \sigma_2(n)}(2)\}$, $\{\alpha_{\sigma_1 \circ \dots \circ \sigma_k(n)}(k)\}$ are Ω -subsequences by Lemma 4.4 (i) and (ii). Let $\sigma = \sigma_1 \circ \dots \circ \sigma_k$ then $\{\alpha_{\sigma(n)}\}$ is an Ω -subsequence of $\{\alpha_n\}$ by Lemma 4.4(iii). \square

Definition 4.6. Let $B = \{p_1, \dots, p_k\}$, $\mathcal{M} = \{\prod_{i=1}^k p_i^{\alpha(i)} : \alpha \in \mathbb{N}^k\}$. $(a \mapsto b) = \{c \in \mathcal{M} : ac \leq b\}$ for all $a \in \mathcal{M}, b \in B$ are as Definition 3.5.

Lemma 4.7. For every $p \in B$, let $M \Rightarrow p = \{(m \mapsto p) : m \in \mathcal{M}\}$. Then (i) $M \Rightarrow p$ is linearly ordered under set inclusion; (ii) $M \Rightarrow p$ is finite.

Proof. (i) Let $(m \mapsto p), (m' \mapsto p) \in M \Rightarrow p$ then $(m \mapsto p) \subseteq (m' \mapsto p)$ if $m' \leq m$ and $(m' \mapsto p) \subseteq (m \mapsto p)$ otherwise. Then $M \Rightarrow p$ is linearly ordered under set inclusion.

(ii) Suppose that there is an infinite ascending sequence $\left\{ \left(\prod_{i=1}^k p_i^{\beta_n(i)} \mapsto p \right) \right\}$ in $M \Rightarrow p$ under set inclusion, i.e., $\beta_n \in \mathbb{N}^k$ and

$$\left(\prod_{i=1}^k p_i^{\beta_n(i)} \mapsto p \right) \subset \left(\prod_{i=1}^k p_i^{\beta_{n+1}(i)} \mapsto p \right)$$

for all $n \in \mathbb{Z}_+$. By Lemma 4.4, there is a subsequence index τ such that $\{\beta_{\tau(n)}\}$ is an Ω -subsequence of $\{\beta_n\}$. Then

$$\left(\prod_{i=1}^k p_i^{\beta_{\tau(n)}(i)} \mapsto p \right) \subset \left(\prod_{i=1}^k p_i^{\beta_{\tau(n+1)}(i)} \mapsto p \right)$$

for all n in \mathbb{Z}_+ , where

$$\beta_{\tau(1)}(i) = \beta_{\tau(2)}(i) = \cdots = \beta_{\tau(n)}(i) = \cdots$$

or

$$\beta_{\tau(1)}(i) < \beta_{\tau(2)}(i) < \cdots < \beta_{\tau(n)}(i) < \cdots$$

for each $1 \leq i \leq k$. Thus there is a sequence $\{\alpha_n\}$ in \mathbb{N}^k such that

$$\prod_{i=1}^k p_i^{\alpha_n(i)} \in \left(\prod_{i=1}^k p_i^{\beta_{\tau(n+1)}(i)} \mapsto p \right) \text{ and } \prod_{i=1}^k p_i^{\alpha_n(i)} \notin \left(\prod_{i=1}^k p_i^{\beta_{\tau(n)}(i)} \mapsto p \right).$$

Thus for all n in \mathbb{Z}_+ ,

$$\prod_{i=1}^k p_i^{\beta_{\tau(n+1)}(i)} \prod_{i=1}^k p_i^{\alpha_n(i)} \leq p < \prod_{i=1}^k p_i^{\beta_{\tau(n)}(i)} \prod_{i=1}^k p_i^{\alpha_n(i)}.$$

By Lemma 4.4, there is a subsequence index σ such that $\{\alpha_{\sigma(n)}\}$ is an Ω -subsequence of $\{\alpha_n\}$. Then

$$\alpha_{\sigma(1)}(i) = \alpha_{\sigma(2)}(i) = \cdots = \alpha_{\sigma(n)}(i) = \cdots$$

or

$$\alpha_{\sigma(1)}(i) < \alpha_{\sigma(2)}(i) < \cdots < \alpha_{\sigma(n)}(i) < \cdots$$

for each $1 \leq i \leq k$. Then for all n in \mathbb{Z}_+ ,

$$\prod_{i=1}^k p_i^{\beta_{\tau(\sigma(n)+1)}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(n)}(i)} \leq p < \prod_{i=1}^k p_i^{\beta_{\tau(\sigma(n))}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(n)}(i)}.$$

Then

$$\prod_{i=1}^k p_i^{\beta_{\tau(\sigma(1)+1)}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(1)}(i)} \leq p < \prod_{i=1}^k p_i^{\beta_{\tau(\sigma(1))}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(1)}(i)},$$

$$\prod_{i=1}^k p_i^{\beta_{\tau(\sigma(3)+1)}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(3)}(i)} \leq p < \prod_{i=1}^k p_i^{\beta_{\tau(\sigma(3))}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(3)}(i)}.$$

Thus

$$\prod_{i=1}^k p_i^{\beta_{\tau(\sigma(1)+1)}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(1)}(i)} < \prod_{i=1}^k p_i^{\beta_{\tau(\sigma(3))}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(3)}(i)}$$

and

$$\prod_{i=1}^k p_i^{\beta_{\tau(\sigma(3)+1)}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(3)}(i)} < \prod_{i=1}^k p_i^{\beta_{\tau(\sigma(1))}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(1)}(i)}.$$

Since

$$\sigma(1) + 1 \leq \sigma(2) < \sigma(3), \sigma(1) < \sigma(3) + 1$$

then

$$\tau(\sigma(1) + 1) < \tau(\sigma(3)), \tau(\sigma(1)) < \tau(\sigma(3) + 1).$$

Thus for all $1 \leq i \leq k$,

$$\beta_{\tau(\sigma(3))}(i) - \beta_{\tau(\sigma(1)+1)}(i) \geq 0,$$

$$\beta_{\tau(\sigma(3)+1)}(i) - \beta_{\tau(\sigma(1))}(i) \geq 0,$$

$$\alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i) \geq 0.$$

Hence

$$e < \prod_{i=1}^k p_i^{\beta_{\tau(\sigma(3))}(i) - \beta_{\tau(\sigma(1)+1)}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i)}$$

and

$$\prod_{i=1}^k p_i^{\beta_{\tau(\sigma(3)+1)}(i) - \beta_{\tau(\sigma(1))}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i)} < e.$$

Since for all $1 \leq i \leq k$,

$$\beta_{\tau(\sigma(3))}(i) - \beta_{\tau(\sigma(1)+1)}(i) > 0$$

iff

$$\beta_{\tau(\sigma(3)+1)}(i) - \beta_{\tau(\sigma(1))}(i) > 0$$

by

$$\tau(\sigma(1)) < \tau(\sigma(1) + 1) < \tau(\sigma(3)) < \tau(\sigma(3) + 1),$$

then

$$\beta_{\tau(\sigma(3))}(i) - \beta_{\tau(\sigma(1)+1)}(i) + \alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i) > 0$$

iff

$$\beta_{\tau(\sigma(3)+1)}(i) - \beta_{\tau(\sigma(1))}(i) + \alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i) > 0.$$

Hence

$$\prod_{i=1}^k p_i^{\beta_{\tau(\sigma(3))}(i) - \beta_{\tau(\sigma(1)+1)}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i)} > e$$

if and only if

$$\prod_{i=1}^k p_i^{\beta_{\tau(\sigma(3)+1)}(i) - \beta_{\tau(\sigma(1))}(i)} \prod_{i=1}^k p_i^{\alpha_{\sigma(3)}(i) - \alpha_{\sigma(1)}(i)} > e$$

by Lemma 2.9, which is a contradiction. Hence there is no infinite ascending sequence in $M \Rightarrow p$. Similarly, we proved that there is no infinite descending sequence in $M \Rightarrow p$. Thus $M \Rightarrow p$ is finite by (i). \square

Lemma 4.8. *If \mathcal{B} is a finite partial subalgebra of \mathcal{A} then the algebra \mathcal{D} is finite.*

Proof. It is immediately from Lemma 4.7. \square

Theorem 4.9. *The varieties of \mathbf{UL}_ω -algebras and \mathbf{IUL}_ω -algebras have the FEP.*

Proof. It is immediately from Lemma 3.9~3.11, Lemma 4.8. \square

Theorem 4.10. *Let $\mathbf{L} \in \{\mathbf{UL}, \mathbf{IUL}\}$ and \mathbf{L}_ω denote \mathbf{L} plus (FIN). For any formula φ in \mathbf{L}_ω , the following statements are equivalent:*

- (i) $\Gamma \vdash_{\mathbf{L}_\omega} \varphi$;
- (ii) $\Gamma \models_{\mathcal{A}} \varphi$ for every \mathbf{L}_ω -algebra \mathcal{A} ;
- (iii) $\Gamma \models_{\mathcal{A}} \varphi$ for every \mathbf{L}_ω -chain \mathcal{A} ;
- (iv) $\Gamma \models_{\mathcal{A}} \varphi$ for every finite \mathbf{L}_ω -algebra \mathcal{A} ;
- (v) $\Gamma \models_{\mathcal{A}} \varphi$ for every finite \mathbf{L} -algebra \mathcal{A} .

Proof. (iii) implies (ii) by Theorem 2.7. Clearly, (ii) implies (iv). Then (iii) implies (vi). (vi) implies (iii) by Theorem 4.9. (iv) is equivalent to (v) by Lemma 2.12 (ii). \square

Corollary 4.11. *The universal theories of \mathbf{UL}_ω -algebras and \mathbf{IUL}_ω -algebras are decidable.*

5. Wang's Construction and Standard completeness

In this section, let $\mathbf{L}_\omega \in \{\mathbf{HpsUL}_\omega^*, \mathbf{UL}_\omega, \mathbf{IUL}_\omega\}$, $\mathcal{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, \rightsquigarrow, e, f, \perp, \top \rangle$ be a finite or countable linearly ordered \mathbf{L}_ω -algebra and s, t, u be arbitrary elements of A . We prove that \mathbf{L}_ω is standard complete by Wang's constructions [13~15], which are some generalizations of Jenei and Montagna-style approach for proving standard completeness for monoidal t-norm based logic **MTL** [8] and its extensions [5].

Definition 5.1. [13, 14] Let \mathcal{A} be an \mathbf{HpsUL}_ω^* -algebra (or an \mathbf{UL}_ω -algebra). For each $s \in A$, t is the immediate predecessor of s in A if (i) $t \in A$, $t < s$; (ii) $\forall u \in A, u < s$ implies $u \leq t$. For each $s \in A$, let s^- denote the immediate predecessor of s in A if it exists, otherwise take $s^- = s$.

Let $X = \{(s, 1) : s \in A\} \cup \{(s, q) : s \in A, s > s^-, q \in Q \cap (0, 1)\}$, we define:
 $(s, q) \leq (t, r)$ iff either $s <_S t$, or $s = t$ and $q \leq r$ and,

$$\begin{aligned} I_1 &:= \{(s, t) : s, t \in A, st = s \neq t, s > s^-t\} \\ I_2 &:= \{(s, t) : s, t \in A, st = t \neq s, t > st^-\} \\ I_3 &:= \{(s, t) : s, t \in A, st = t = s, s > st^-\} \\ I_4 &:= \{(s, t) : s, t \in A, (st \neq t \text{ and } st \neq s) \text{ or } \\ &\quad (st = s^-t = s) \text{ or } (st = st^- = t)\}. \end{aligned}$$

Now define, for $(s, q), (t, r) \in X$:

$$(s, q) \circ (t, r) = \begin{cases} (s, q) & (s, t) \in I_1, \\ (t, r) & (s, t) \in I_2, \\ (s, q) \wedge_X (t, r) & (s, t) \in I_3, \\ (st, 1) & (s, t) \in I_4, \end{cases}$$

where \wedge_X and \vee_X is meant \min_X and \max_X with respect to \leq_X , respectively. We will omit index if it does not cause confusion.

Lemma 5.2. *Let \mathcal{A} be an \mathbf{HpsUL}_ω^* -algebra (or an \mathbf{UL}_ω -algebra). Then $(s, q) \circ (t, r) \leq (e, 1)$ iff $(s, q) \circ (t, r) \circ (t, r) \leq (e, 1)$ for all $(s, q), (t, r)$ in X .*

Proof. Let $(s, q) \circ (t, r) \leq (e, 1)$. Then $st \leq e$, since $(s, q) \circ (t, r) = (st, \diamond)$ for some $\diamond \in \{q, r, 1\}$ by Definition 5.1. Thus $stt \leq e$ by (Fin). Hence $(s, q) \circ (t, r) \circ (t, r) \leq (e, 1)$. The sufficiency part of the lemma can be proved in the same way. \square

Definition 5.3. [15] Let \mathcal{A} be an \mathbf{IUL}_ω -algebra. Let

$$I^* := \{(s, t) : s, t \in A, s^- < s, t^- < t, t = \neg s^-\},$$

$$I^{**} := \{(s, t) : s, t \in A, ss = s^-s = s = t\}.$$

$\forall (s, q), (t, r) \in X$, define

$$(s, q) \triangle (t, r) = \begin{cases} (s, q) \circ (t^-, 1) \vee (s^-, 1) \circ (t, r) & \text{if } (s, t) \in I^*, q + r \leq 1, \\ (s, q \vee r) \circ (s^-, 1) & \text{if } (s, t) \in I^{**}, \\ (s, q) \circ (t, r) & \text{otherwise.} \end{cases}$$

Lemma 5.4. Let \mathcal{A} be an \mathbf{IUL}_ω -chain and $s, t \in A$. (i) If $st^- \neq s$, $st^- \leq e$, $s^-t \leq e$ then $st^-t \leq e$; (ii) If $st^- = s^-t^-$ and $s^-t \leq e$ then $st^-t \leq e$; (iii) $(s, q) \triangle (t, r) \leq (s, q) \circ (t, r)$.

Proof. (i) If $st \leq e$ then $stt \leq e$ by Lemma 2.9 and thus $st^-t \leq stt \leq e$. If $t \leq e$ then $st^-t \leq t \leq e$ by $st^- \leq e$. Thus, let $st > e$ and $t > e$ in the following.

$t^- \geq e$ by $t > e$. $t^- \neq e$ by $st^- \neq s$. Then $t^- > e$. Thus $st^- \geq s$. Hence $st^- > s$ by $st^- \neq s$. $st^- \neq e$ by Lemma 11(4) and $t^- > e$. Therefore $st^- < e$ by $st^- \leq e$. Then $st^- < e < t^-$. Thus $st^- < t^-$. Hence $s < e$.

Suppose that $st \leq t^-$. Then $sst \leq st^- \leq e$. Thus $st \leq e$ by Lemma 2.9, a contradiction and hence $st > t^-$. Therefore $st \geq t$. $st \leq t$ by $s < e$. Then $st = t$.

Suppose that $s^-t \geq s$ then $s^-tt \geq st > e$. Thus $s^-t > e$ by Lemma 2.9, a contradiction and hence $s^-t < s$.

Therefore $s^-t \leq s^-$. $s^-t \geq s^-$ by $t > e$. Then $s^-t = s^-$. Then $s^-st = s^-$ by $st = t$. Thus $s^-s = s^-$ by Lemma 2.11(1).

Suppose that $ss = s$ then $st^- = sst^- \leq s$, a contradiction with $st^- > s$ and hence $ss < s$ by $ss \leq s$. Then $ss \leq s^-$.

Thus $s^- = s^-s \leq ss \leq s^-$. Hence $ss = s^-$. Then $(ss)t = s^-t = s^-$ and $s(st) = st = t$. Thus $s^- = t$ by $(ss)t = s(st)$, a contradiction with $s^- < e < t$. Thus the case of $st > e$ and $t > e$ does not exist. This completes the proof of (i).

(ii) It follows from $s^-t \leq e$ that $s^-tt \leq e$ by Lemma 2.9. Then $st^-t = s^-t^-t \leq s^-tt \leq e$ by $st^- = s^-t^-$ and thus $st^-t \leq e$.

(iii) see Proposition 3.7 (2) of [15]. \square

Lemma 5.5. Let \mathcal{A} be a finite \mathbf{IUL}_ω -algebra. Then $(s, q) \triangle (t, r) \leq (e, 1)$ if and only if $(s, q) \triangle (t, r) \triangle (t, r) \leq (e, 1)$ for all $(s, q), (t, r)$ in X .

Proof. Let $(s, q) \Delta (t, r) \leq (e, 1)$. There are three cases to be considered.

Case 1. $(s, t) \in I^*$ and $q + r \leq 1$. Then $(s, q) \Delta (t, r) = (s, q) \circ (t^-, 1) \vee (s^-, 1) \circ (t, r) \leq (e, 1)$. Thus $st^- \leq e$, $s^-t \leq e$. Then $s^-tt \leq e$ by Lemma 2.9. If $(s, q) \Delta (t, r) = (s^-, 1) \circ (t, r)$ then $(s, q) \Delta (t, r) \Delta (t, r) = ((s^-, 1) \circ (t, r)) \Delta (t, r) \leq ((s^-, 1) \circ (t, r)) \circ (t, r) \leq (s^-tt, 1) \leq (e, 1)$ by Lemma 5.4(iii). Let $(s, q) \Delta (t, r) = (s, q) \circ (t^-, 1)$ in the following. If $(s, q) \circ (t^-, 1) = (s, q)$ then $(s, q) \Delta (t, r) \Delta (t, r) = (s, q) \Delta (t, r) \leq (e, 1)$. Otherwise $st^- \neq s$ or $st^- = s^-t^-$. Then $st^-t \leq e$ by Lemma 5.4. Thus $(s, q) \Delta (t, r) \Delta (t, r) = ((s, q) \circ (t^-, 1)) \Delta (t, r) \leq ((s, q) \circ (t^-, 1)) \circ (t, r) \leq (st^-t, 1) \leq (e, 1)$.

Case 2. $(s, t) \in I^{**}$ then $ss = s^-s = s = t$ and $(s, q) \Delta (t, r) = (s, q \vee r) \circ (s^-, 1) \leq (e, 1)$. Thus $ss^- \leq e$. Hence $ss^-s \leq e$ by Lemma 2.9. Therefore $(s, q) \Delta (t, r) \Delta (t, r) = ((s, q \vee r) \circ (s^-, 1)) \Delta (s, r) \leq ((s, q \vee r) \circ (s^-, 1)) \circ (s, r) \leq (ss^-s, 1) \leq (e, 1)$.

Case 3. $(s, q) \Delta (t, r) = (s, q) \circ (t, r) \leq (e, 1)$ then $st \leq e$. Thus $stt \leq e$ by Lemma 2.9. Hence, by Lemma 5.4(iii), $(s, q) \Delta (t, r) \Delta (t, r) \leq (s, q) \circ (t, r) \circ (t, r) \leq (stt, 1) \leq (e, 1)$.

By a similar procedure, we prove that $(s, q) \Delta (t, r) \leq (e, 1)$ if $(s, q) \Delta (t, r) \Delta (t, r) \leq (e, 1)$. \square

Lemma 5.6. *Let \mathcal{A} be an \mathbf{HpsUL}_ω^* -algebra, X and the binary operation \circ on X be as in Definition 5.2. The following conditions hold:*

- (a) X is densely ordered, and has a maximum $\top_X = (\top, 1)$ and a minimum $\perp_X = (\perp, 1)$.
- (b) $\langle X, \circ, \leq_X, e_X \rangle$ is a linearly ordered monoid, where $e_X = (e, 1)$.
- (c) \circ is left-continuous with respect to the order topology on $\langle X, \leq_X \rangle$.
- (d) There is a map Φ from A into X such that Φ is an embedding of the structure $\langle A, \wedge, \vee, \cdot, e, \perp, \top \rangle$ into $\langle X, \wedge_X, \vee_X, \circ, e_X, \perp_X, \top_X \rangle$, and for all $s, t \in A$, $\Phi(s \rightarrow t)$, $\Phi(s \rightsquigarrow t)$ are the right and left residua of $\Phi(s)$ and $\Phi(t)$ in $\langle X, \wedge_X, \vee_X, \circ, e_X, \perp_X, \top_X \rangle$, respectively.
- (e) $\forall (s, q), (t, r) \in X, (s, q) \circ (t, r) \leq (e, 1)$ iff $(s, q) \circ (t, r) \circ (t, r) \leq (e, 1)$.
- (f) $(s, q) \circ (t, r) \leq (e, 1)$ implies $(t, r) \circ (s, q) \leq (e, 1)$ for any $(s, q), (t, r)$ in X .

Proof. Claim (e) has been proved by Lemma 5.2. As pointed out in [13], the associativity of \circ is mainly dependent on Lemma 2.11 (1)~(3). Then other claims can be proved in the same way as that of [13, Theorem 4.5]. \square

Lemma 5.7. *Every countable linearly ordered \mathbf{HpsUL}_ω^* -algebra can be embedded into a standard \mathbf{HpsUL}_ω^* -algebra.*

Proof. Let X, \mathcal{A} , etc. be as in Definition 5.1. We can assume, without loss of generality, that $X = Q \cap [0, 1]$. Now define for $\alpha, \beta \in [0, 1]$, $\alpha * \beta = \sup\{x \circ y : x, y \in X, x \leq \alpha, y \leq \beta\}$. The proof of the weak commutativity, the monotonicity, associativity, left-continuity, etc. of $*$ is the same as that of [13, Theorem 4.6]. The neutral element of $*$ is e_X in $Q \cap [0, 1]$. By the left-continuity of $*$, the following property holds.

(P) $\alpha, \beta, \gamma \in [0, 1]$, $\alpha * \beta * \gamma = \sup\{x \circ y \circ z : x, y, z \in X, x \leq \alpha, y \leq \beta, z \leq \gamma\}$.

We prove that $\alpha * \beta \leq e_X$ iff $\alpha * \beta * \beta \leq e_X$ for any α, β in $[0, 1]$. Given $\alpha * \beta \leq e_X$ then $x \circ y \leq e_X$ for all $x, y \in X, x \leq \alpha, y \leq \beta$. Let $x, y, z \in X, x \leq \alpha, y \leq \beta, z \leq \beta$. Then $x \circ y \leq e_X, x \circ z \leq e_X$. Thus $x \circ y \circ y \leq e_X, x \circ z \circ z \leq e_X$ by Lemma 5.6(e). Hence $x \circ y \circ z \leq x \circ y \circ y \vee x \circ z \circ z \leq e_X$. Therefore $\alpha * \beta * \beta \leq e_X$ by (P). The sufficient part of the claim can be proved in the similar way. \square

By Lemma 2.11, Definition 5.3, Lemma 5.5, we can prove the claims similar to Lemma 5.6 and 5.7 for \mathbf{UL}_ω and \mathbf{IUL}_ω -algebras. As a consequence of these lemmas, and extending [8, Theorem 3.3] in the obvious way, we obtain the following standard completeness.

Theorem 5.8. *\mathbf{HpsUL}_ω^* , \mathbf{UL}_ω and \mathbf{IUL}_ω are complete with respect to the class of standard algebras involved.*

6. Concluding remarks

Theorem 4.10 shows that, as was expected, axiomatic systems \mathbf{UL}_ω and \mathbf{IUL}_ω are complete with respect to finite \mathbf{UL} and \mathbf{IUL} -algebras, respectively. The suitability of Blok and Alten's Construction for \mathbf{UL}_ω , \mathbf{IUL}_ω -algebras mainly depends on that elements of the monoid M generated by $\{p_1, \dots, p_k\}$ has the form $\prod_{i=1}^k p_i^{\alpha(i)}$. It seems difficult to extend the proof of Lemma 4.7 to \mathbf{HpsUL}_ω^* -algebras.

References

- [1] W. J. Blok and C. J. Alten, The finite embeddability property for residuated lattices, pocrimis and BCK-algebras, *Algebra univers.* **48** (2002) 253-271.
- [2] W. J. Blok and C. J. Alten, On the finite embeddability property for residuated ordered groupoids. *Transactions of the American Mathematical Society*, **357**(10)(2005) 4141-4157.

- [3] A. Ciabattoni, G. Metcalfe, and F. Montagna. Adding modalities to MTL and its extensions. Proceedings of the Linz Symposium 2005, 2005.
- [4] P. Cintula, C. Noguera, Implicational (semilinear) logics I: a new hierarchy, Arch. Math. Log. **49**(2010), 417-446.
- [5] F. Esteva, J. Gispert, L. Godo, and F. Montagna. On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic, Studia Logica, **71**(2)(2002) 199-226.
- [6] Z.Hanikova and R. Horcik. The finite embeddability property for residuated groupoids, Algebra universalis **72**(1) (2014) 1-13.
- [7] R. Horcik. Finite Embeddability Property for Residuated Lattices via Regular Languages, available online, <http://www2.cs.cas.cz/~horcik/preprints/fep.pdf>
- [8] S. Jenei and F. Montagna. A proof of standard completeness for Esteva and Godo's logic MTL, Studia Logica, **70**(2002)183-192.
- [9] P. Jipsen and C. Tsinakis. A survey of residuated lattices, in Ordered algebraic structures, Springer US, 19-56, 2002.
- [10] G. Metcalfe, N. Olivetti and D. Gabbay. Proof Theory for Fuzzy Logics, Springer Series in Applied Logic (Vol.36), 2009.
- [11] G. Metcalfe and F. Montagna. Substructural fuzzy logics, *Journal of Symbolic Logic*, **7**(3), 834-864, 2007.
- [12] C. Tsinakis and K. Blount, The structure of residuated lattices, Int. j. algebra. Comput. **13**(4)(2003)437-461.
- [13] S.-M. Wang, Logics for residuated pseudo-uninorms and their residua, Fuzzy Sets and Systems, **218** (2013) 24-31.
- [14] S.-M. Wang. Uninorm logic with the n -potency axiom, Fuzzy Sets and Systems, **205**(2012)116-126.
- [15] S.-M. Wang. Involutive uninorm logic with the n -potency axiom, Fuzzy Sets and Systems, **218** (2013) 1-23.
- [16] S.-M. Wang. The Finite Model Property for Semilinear Substructural Logics, Mathematical Logic Quarterly, **59** (4-5)(2013)268-273.